THE GLOBAL ORBIT ∞-CATEGORY AND APPLICATIONS TO ASSEMBLY MAPS

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Contents

1. Introduction

This is a sneak peek of my master's thesis, due on May 1st, 2026, and advised by Marco Varisco at the University at Albany, State University of New York.

One of the main protagonists of this thesis is the *global orbit ∞-category of discrete groups Orb*. It is the ∞-category obtained from the $(2,1)$-category whose objects are all discrete groups, 1-morphisms are injective group homomorphisms, and 2-morphisms are given by conjugation; see Definition 5.2. We can take the slice of this ∞-category over any fixed discrete group G, and by the following theorem we recover the classical *orbit 1-category* $\mathcal{O}(G)$, whose objects are the transitive G-sets G/H where H is a subgroup of G and whose morphisms are G-equivariant functions; see Definition 5.1.

Theorem (Theorem 5.3)**.** *For each discrete group G, the ∞-categorical slice $Orb_{/G}$ is equivalent to the 1-category $\mathcal{O}(G)$.*

The analogue of this theorem for *compact Lie* groups was asserted without proof in an unpublished preprint by Gepner and Henriques [GH07], where global orbit categories were first introduced. For compact Lie groups, a proof sketch appears in an unpublished preprint by Rezk [Rez14, Example 3.5.1], and a published proof is

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given by Linksens, Nardin, and Pol in [LNP25, Lemma 6.12]. We give a detailed and somewhat simplified proof for arbitrary discrete groups in Section 5.

Using the theorem above we easily deduce the following corollary, a version of which appears in [LRV03, Lemma 3.11]. Here, $\mathcal{S}(G)$ is the 1-category whose objects are the subgroups H of G, and whose morphisms are equivalence classes of group homomorphisms $f\colon H \to K$ given by conjugation by some element $g \in G$, where two homomorphisms are equivalent if they differ by conjugation by some element of K. We also have a canonical functor $\mathcal{O}(G) \to \mathcal{S}(G)$ sending G/H to H; see Definition 6.1 for details.

Corollary (Corollary 6.3)**.** *Let $\mathcal{D}$ be an ∞-category equivalent to the nerve of a 1-category and let $X : Orb \to \mathcal{D}$ be a functor. For each discrete group G, the composition*

$$\mathcal{O}(G) \simeq Orb_{/G} \xrightarrow{\pi} Orb \xrightarrow{X} \mathcal{D}$$

factors through the canonical functor $\mathcal{O}(G) \to \mathcal{S}(G)$.

Now fix a functor $X\colon Orb \to \mathcal{D}$ to a cocomplete ∞-category $\mathcal{D}$, and fix a family $\mathcal{F}$, i.e., a class of groups closed under isomorphisms and subgroups. Then, for each discrete group G, we can define the *assembly map*

$$\alpha_G^{\mathcal{F}}\colon \operatorname*{colim}_{Orb^{\mathcal{F}}_{/G}} X \to X(G)$$

where $Orb^{\mathcal{F}}_{/G}$ is the slice ∞-category of the full subcategory $Orb^{\mathcal{F}} \subseteq Orb$ spanned by all groups in the family $\mathcal{F}$; see Section 8 for details. Given nested families $\mathcal{F}' \subseteq \mathcal{F}$, we have the following commutative triangle

$$\begin{array}{ccc} \operatorname*{colim}_{Orb^{\mathcal{F}'}_{/G}} X & & \\ & \searrow^{\alpha_G^{\mathcal{F}'}} & \\ \downarrow{\scriptstyle \alpha_G^{\mathcal{F}',\mathcal{F}}} & & X(G) \\ & \nearrow_{\alpha_G^{\mathcal{F}}} & \\ \operatorname*{colim}_{Orb^{\mathcal{F}}_{/G}} X & & \end{array}$$

where $\alpha_G^{\mathcal{F}',\mathcal{F}}$ is called the relative assembly map. It is then natural to ask when the relative assembly map is an equivalence. This is answered by the following theorem, called the *Transitivity Principle*, and originally proved by Lück and Reich in [LR05, Theorem 65 (and Lemma 153)] in the context of equivariant homology theories. For functors X out of the global orbit ∞-category, we use the theorem and the setup above to give a more conceptual and categorical proof.

Theorem (Transitivity Principle; Theorem 8.7)**.** *Let X, $\mathcal{D}$, $\mathcal{F}$, and G be as above. The relative assembly map*

$$\alpha_G^{\mathcal{F}',\mathcal{F}}\colon \operatorname*{colim}_{Orb^{\mathcal{F}'}_{/G}} X \to \operatorname*{colim}_{Orb^{\mathcal{F}}_{/G}} X$$

is an equivalence if, for each subgroup H of G with $H \in \mathcal{F}$, the assembly map

$$\alpha_H^{\mathcal{F}'}\colon \operatorname*{colim}_{Orb^{\mathcal{F}'}_{/H}} X \to X(H)$$

is an equivalence.

Next, we turn to examples. Assembly maps for algebraic K-theory and the related Farrell-Jones Conjecture are intensely studied, and we can re-frame them in our global framework. We show that for any ring spectrum R, the assignment $G \mapsto K(R[G])$ gives a functor $Orb \to Sp$, where Sp is the ∞-category of spectra. We take a general and conceptual approach using the equivalence [CMNN24, Example 2.19]

$$\mathrm{Perf}(R)_{hG} \simeq \operatorname*{colim}_{Orb^{\{1\}}_{/G}} \mathrm{Perf}(R) \xrightarrow{\simeq} \mathrm{Perf}(R[G]),$$

allowing us to apply the construction not only to K, but also to THH, TC, and other similar functors.

...and more to come...

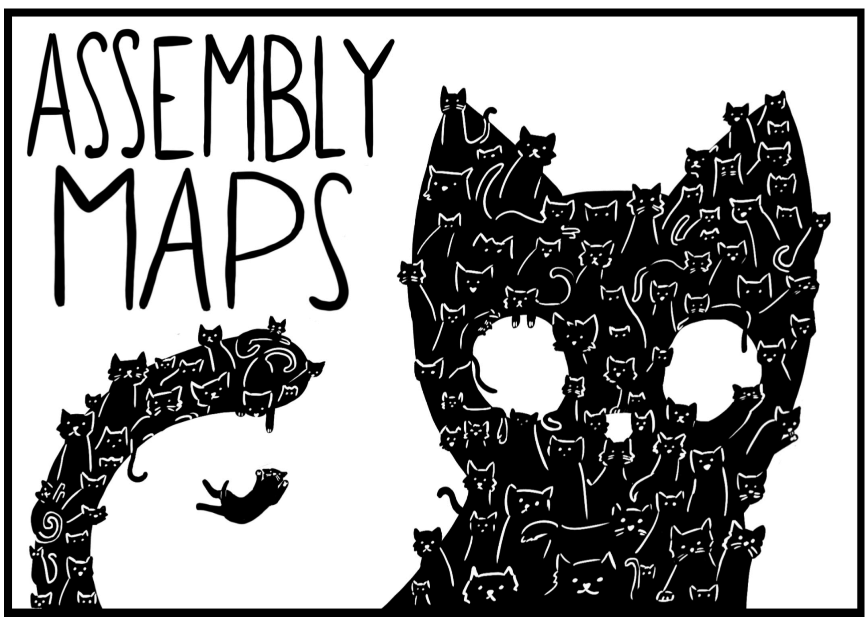

FIGURE 1. Inspired by Maxine Calle's doodles

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